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Synchronization in stochastic coupled systems: theoretical results

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Abstract
The stability of synchronized states (including equilibrium point, periodic orbit or chaotic attractor) in stochastic coupled dynamical systems (ordinary differential equations) is considered. A general approach is presented, based on the master stability function, Gershgörin disc theory and the extreme value theory in statistics, to yield constraints on the distribution of coupling to ensure the stability of synchronized dynamics. Three types of different behaviour: global-stable, exponential-stable and power-stable, are found, depending on the nature of the distribution of the interactions between units. Systems with specific coupling schemes are used as examples to illustrate our general method.

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1. Introduction

Large networks of stochastic coupled dynamical systems that exhibit synchronized static, periodic or chaotic dynamics are subjects of great interest in a variety of fields ranging from biology [10, 16, 17, 35] to semiconductor lasers [29, 30, 33, 39] to electronic circuits [22, 45]. For example, in neuronal systems how to synchronize a group of neurons in the cortex, the so-called 'binding problem', is a fundamental issue. It is relatively easy to work out conditions on how to synchronize a group of neurons without interactions, see for example [31] for experimental results and [11] for theoretical results.

For a given system it is essential to know the extent to which the coupling strengths can be varied so that the synchronized state remains stable. In other words, to generalize results in [11, 31] to models with interactions is a challenging problem. For a system with deterministic interaction, early attempts [3, 4, 6, 7, 12, 14, 23–25, 27, 33, 37, 40, 43] at this question have typically looked either at systems of very small size or at very specific coupling schemes

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(diffusive coupling, global all to all coupling etc with a single coupling strength). Recent work [21, 46] introduced the notion of a master stability function that enables the analysis of general coupling topologies. This function defines a region of stability in terms of the eigenvalues of the coupling matrix. On the other hand, with a specific form of interactions between neurons (Kuramoto dynamics), many interesting results have been obtained [10, 41]. All results mentioned above are for systems with a deterministic interaction between units, although a system with stochastic interactions is of great interest for many purposes, see for example [5, 9, 32]. In particular, in a neuronal system, the actual interaction strengths between neurons remain unclear. Moreover, it is recently reported that the interaction between neurons changes within a time window less than 50 ms. Hence it is probably reasonable to treat neuronal models as a system with stochastic interactions, as demonstrated in the classical Hopfield model.

In this paper we present a general method that provides explicit constraints on the stochastic coupling strengths themselves by combining the master stability function, the Gershgörin disc theory with the extreme value theory in statistics. Commonly studied coupling schemes are used as illustrative examples. We intend to report further applications of our results to spiking neuronal models, small-world networks etc in a future publication.

2. Preliminary results on coupled ODEs [8]

Let us consider a system of coupled ODEs which are represented by

$$\dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i) + \frac{1}{N}\sum_{j=1}^{N} G_{ij}\mathbf{H}(\mathbf{x}^j) \tag{1}$$

where $\mathbf{x}^i$ is the M-dimensional vector of the ith node, $\mathbf{F} = (F^1, F^2, \ldots, F^M) : R^M \to R^M$, $\mathbf{H} = (H^1, H^2, \ldots, H^M) : R^M \to R^M$ is the coupling function. We define $\mathbf{G} = [G_{ij}]$ as the $N \times N$ coupling matrix where G_{ij} gives the stochastic coupling strength from map j to map i. The condition $\sum_{j=1}^{N} G_{ij} = 0, i = 1, \ldots, N$ is imposed to ensure that synchronized dynamics is a solution to equation (1). For example, when $(\mathbf{x}, \ldots, \mathbf{x})$ is a fixed point of the dynamics

$$\dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i)$$

then $(\mathbf{x}, \ldots, \mathbf{x})$ is again a fixed point of the dynamics (1). The dynamics of the individual node (unit) is $\dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i)$. Linearizing around the synchronized state $\mathbf{x}$ we get

$$\dot{\mathbf{z}}^i = \mathbf{J}(\mathbf{x}) \cdot \mathbf{z}^i + \frac{1}{N}\sum_{j=1}^{N} G_{ij} \cdot D\mathbf{H}(\mathbf{x}) \cdot \mathbf{z}^j \tag{2}$$

where $\mathbf{z}^i$ denotes deviations from the stable point $\mathbf{x}$, $\mathbf{J}(\cdot)$ and $D\mathbf{H}(\cdot)$ are the $M \times M$ Jacobian matrices for the functions of $\mathbf{F}$ and $\mathbf{H}$. In terms of the $M \times N$ matrix $\mathbf{S}(t) = (\mathbf{z}^1(t)\mathbf{z}^2(t)\cdots\mathbf{z}^N(t))$, equation (2) can be recast as

$$\dot{\mathbf{S}} = \mathbf{J}(\mathbf{x}) \cdot \mathbf{S} + \frac{1}{N} D\mathbf{H}(\mathbf{x}) \cdot \mathbf{S} \cdot \mathbf{G}^T. \tag{3}$$

According to the theory of Jordan canonical forms, the stability of equation (3) is determined by the eigenvalue λ of $\mathbf{G}$. Let λ be an eigenvalue of $\mathbf{G}^T$. Denote the corresponding eigenvector by $\mathbf{e}$. Let $\mathbf{u}(t) = \mathbf{S}(t)\mathbf{e}$. We obtain

$$\dot{\mathbf{u}} = \left[\mathbf{J}(\mathbf{x}) + \frac{1}{N}\lambda \cdot D\mathbf{H}(\mathbf{x})\right]\mathbf{u}. \tag{4}$$

So the stability problem originally formulated in the $M \times N$ space has been reduced to a problem in a $M \times M$ space where it is often the case that $M \ll N$. It is worth mentioning that this eigenvalue based analysis is valid even if the coupling matrix $\mathbf{G}$ is defective [19].

We note that $\lambda = 0$ is always an eigenvalue of $\mathbf{G}$ and its corresponding eigenvector is $(11\cdots 1)^T$ due to the synchronization constraint $\sum_{j=1}^{N} G_{ij} = 0$. In this case, equation (4) can be used to generate the Lyapunov exponents for the individual system $\dot{\mathbf{u}}^i = \mathbf{J}(\mathbf{x})\mathbf{u}$, which we denote by $h_1 = h_{\max} \geqslant h_2 \geqslant \cdots \geqslant h_M$. These exponents describe the dynamics within the synchronization manifold defined by $\mathbf{x}^i = \mathbf{x}\ \forall i$.

The subspace spanned by the remaining eigenvectors is transverse to the synchronization manifold, the dynamics in it will be stable if the transverse Lyapunov exponents are all negative. To examine this problem, we treat λ in equation (4) as a complex parameter and calculate the maximum Lyapunov exponent $\mu_{\max}$ as a function of λ. This function is referred to as the master stability function by Pecora and Carroll [36]. The region in the $(\mathrm{Re}(\lambda), \mathrm{Im}(\lambda))$ plane where $\mu_{\max} < 0$ defines a stability zone denoted by Ω. There are two possible configurations of Ω. Whether Ω is an unbounded area or a bounded one is contingent on the coupling scheme and other system parameters. The origin, which is the zero eigenvalue of $\mathbf{G}$, may or may not lie in the stability zone. For example, for equilibrium or periodic state in coupled maps, the origin is in Ω, but for chaos, it lies outside Ω. We note that, typically, Ω is obtained numerically. In some instances analytical results are possible (see below).

Clearly, if all the transverse eigenvalues of $\mathbf{G}$ lie within Ω, then the synchronized state is stable. Here we seek constraints applicable directly to the coupling strengths. This problem is dealt with by combining the master stability function with the Gershgörin disc theory.

The Gershgörin disc theorem [20] states that all the eigenvalues of a $N \times N$ matrix $\mathbf{A} = [a_{ij}]$ are located in the union of N discs (called Gershgörin discs) where each disc is given by

$$G_i = \left\{ z \in C : |z - a_{ii}| \leqslant \sum_{j \neq i} |a_{ji}| \right\} \qquad i = 1, 2, \ldots, N. \tag{5}$$

To apply this theorem to the transverse eigenvalues we need to remove $\lambda = 0$. We appeal to an order reduction technique in matrix theory [15] which leads to a $(N-1) \times (N-1)$ matrix $\mathbf{D}$ whose eigenvalues are the same as the eigenvalues of $\mathbf{G}$ except for $\lambda = 0$.

Suppose that, for a given matrix $\mathbf{G}$, we have knowledge of one of its eigenvalues $\tilde{\lambda}$ and the eigenvector $\mathbf{e}$. Through proper normalization we can make any component of $\mathbf{e}$ equal one. Here, without loss of generality, we assume that the first component is made equal 1, namely, $\mathbf{e} = \left(1, \mathbf{e}_{N-1}^T\right)^T$. Rewrite $\mathbf{G}$ in the following block form:

$$\mathbf{G} = \begin{pmatrix} G_{11} & \mathbf{r}^T \\ \mathbf{s} & \mathbf{G}_{N-1} \end{pmatrix} \tag{6}$$

with $\mathbf{r} = (G_{12}, \ldots, G_{1N})^T$, $\mathbf{s} = (G_{21}, \ldots, G_{N1})^T$ and

$$\mathbf{G}_{N-1} = \begin{pmatrix} G_{22} & \cdots & G_{2N} \\ \vdots & \vdots & \vdots \\ G_{N2} & \cdots & G_{NN} \end{pmatrix}. \tag{7}$$

Choose a matrix $\mathbf{P}$ in the form

$$\mathbf{P} = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{e}_{N-1} & \mathbf{I}_{N-1} \end{pmatrix}. \tag{8}$$

Here $\mathbf{I}_{N-1}$ is the $(N-1) \times (N-1)$ identity matrix. We know that

$$\mathbf{P}^{-1} = \begin{pmatrix} 1 & \mathbf{0}^T \\ -\mathbf{e}_{N-1} & \mathbf{I}_{N-1} \end{pmatrix}.$$

Similarity transformation of $\mathbf{G}$ by $\mathbf{P}$ yields

$$\mathbf{P}^{-1}\mathbf{G}\mathbf{P} = \begin{pmatrix} \tilde{\lambda} & \mathbf{r}^T \\ \mathbf{0} & \mathbf{G}_{N-1} - \mathbf{e}_{N-1}\mathbf{r}^T \end{pmatrix}. \tag{9}$$

Since $\mathbf{P}^{-1}\mathbf{G}\mathbf{P}$ and $\mathbf{G}$ have identical eigenvalue spectra, the $(N-1) \times (N-1)$ matrix

$$\mathbf{D}^1 = \mathbf{G}_{N-1} - \mathbf{e}_{N-1}\mathbf{r}^T \tag{10}$$

assumes the eigenvalues of $\mathbf{G}$ are $\tilde{\lambda}$. We can obtain N different versions of the reduced matrix, which we denote by $\mathbf{D}^k (k = 1, 2, \ldots, N)$, depending on which component of $\mathbf{e}$ is made equal 1.

Applying the above technique to the coupling matrix G by letting $\tilde{\lambda} = 0$ and $\mathbf{e} = (11\ldots1)^T$ we get $\mathbf{D}^k = \left[d_{ij}^k\right]$ where $d_{ij}^k = G_{ij} - G_{kj}$. From the Gershgörin theorem the stability conditions of the synchronized dynamics are expressed as

(i) The centre of every Gershgörin disc of $\mathbf{D}^k$ lies inside the stability zone Ω. That is, $(G_{ii} - G_{ki}, 0) \in \Omega$.

(ii) The radius of every Gershgörin disc of $\mathbf{D}^k$ satisfies the inequality

$$\sum_{j=1, j\neq i}^{N} \frac{|G_{ji} - G_{ki}|}{N} \leqslant \delta\left(\frac{G_{ii} - G_{ki}}{N}\right) \qquad i = 1, 2, \ldots, N \quad \text{and} \quad i \neq k. \tag{11}$$

Here $\delta(x)$ is the distance from point x on the real axis to the boundary of the stability zone Ω.

(iii) As k varies from 1 to N, we obtain N sets of stability conditions of equation (11). Each one provides sufficient conditions constraining the coupling strengths.

3. Main results

Now we turn to the stochastic case. We see that the following inequality:

$$\frac{\sum_{j=1, j\neq i}^{N} |G_{ji} - G_{ki}|}{N} - \delta\left(\frac{G_{ii} - G_{ki}}{N}\right) \leqslant 0 \tag{12}$$

implies equation (11). For simplicity of notation we further assume that $G_{ij}, i \neq j$ are an independently identical distribution (i.i.d) random array. Hence

$$\frac{\sum_{j=1, j\neq i}^{N} |G_{ji} - G_{ki}|}{N} \to \langle |G_{ji} - G_{ki}||G_{ki}\rangle \tag{13}$$

and

$$\frac{G_{ii}}{N} \to -\langle G_{12}\rangle \tag{14}$$

as N is large, where $\langle\cdot|\cdot\rangle$ is the conditional expectation. As a consequence of equations (13), (14) and (11), we conclude that

$$\min_k \left[\max_i \left(\langle |G_{ji} - G_{ki}||G_{ki}\rangle - \delta\left(-\langle G_{12}\rangle - \frac{G_{ki}}{N}\right)\right)\right] \leqslant 0 \tag{15}$$

is a sufficient condition to ensure that the synchronized state is stable. Assume that the distribution density of G_{ij} is $p(x)$, we have

$$\eta_{ki} =: \langle |G_{ji} - G_{ki}||G_{ki}\rangle = \int |x - G_{ki}|p(x)\,\mathrm{d}x.$$

Note that $\langle |G_{ji} - G_{ki}||G_{ki}\rangle$ could be infinite. After neglecting the higher order term $\frac{G_{ki}}{N}$, we see that equation (15) can be strengthened requiring

$$\min_k \left[\max_i \langle |G_{ji} - G_{ki}||G_{ki}\rangle\right] < \delta(-\langle G_{12}\rangle).$$

The left-hand side of the inequality above is independent of j and the right-hand side of it is independent of k, i. Next we consider the behaviour of η_{ki}. There are three different types of behaviour, corresponding to three types of distribution in the extremal value theory.

(i) *Global-stable*. This corresponds to the case of type III of the extreme value theory [28]. Essentially, a random variable is type III if it is a bounded random variable. In this case, we assume that $p(x)$ has a compact support set, i.e. G_{ij} is bounded. Thus η_{ki} is also bounded[5]. Hence $\min_k \max_i \eta_{ki}$ is again a bounded variable. The synchronization state is stable if

$$\min_k \max_i \eta_{ki} < \delta(-\langle G_{12}\rangle)$$

and the stability is independent of the system size N.

(ii) *Exponential-stable*. This case is the type I distribution of the extreme value theory. Intuitively, a random variable is of type I if its distribution density is of short tails. As before, we only assume that if $G_{ki}, k \neq i$ is of short-tailed distribution, so is η_{ki}. Let us see an example. Suppose that ξ, η are i.i.d. and obey the exponential distribution with parameter λ, then

$$\begin{aligned}\langle |\xi - \eta||\eta\rangle &= \int_0^{+\infty} |x - \eta|\lambda\,\mathrm{e}^{-\lambda x}\,\mathrm{d}x \\ &= \eta - \frac{1}{\lambda} + \frac{2}{\lambda}\,\mathrm{e}^{-\lambda\eta} \\ &\approx \frac{1}{\lambda} - \eta + \lambda\eta^2 + o(\eta^2).\end{aligned}$$

Therefore the conditional expectation $\langle |\xi - \eta||\eta\rangle$ also is exponentially distributed. We then have by the extreme value theorem for $x > 0$

$$P(\max_i \eta_{ki} - \log N \leqslant x) \to \exp(\mathrm{e}^{-x}) \qquad \text{as} \quad N \to \infty.$$

Thus

$$\max_i \eta_{ki} \sim \log N.$$

Therefore the Gershgörin disc increases at an order of $\log N$. The system is stable under the condition that

$$N < C\exp(\delta(-\langle G_{12}\rangle))$$

where C is a positive constant (see the following section for exact calculations).

[5] We have the following conjecture: if G_{ij} belong to type III, then η_{ki} also belongs to type III. For example, let G_{ij} be the uniform distribution on (0, 1), then $\eta_{ki} = G_{ki}^2 - G_{ki} + \frac{1}{2}$ belongs to type III. A detailed proof is outside the scope of the current paper.

(iii) *Power-stable*. This case is the type II distribution of the extreme value theory. For example, if we assume that G_{ij} is distributed according to the Pareto distribution (long-tail distribution), i.e. the distribution density is $p(x) = \alpha K x^{-\alpha-1}$, $x \geqslant K^{1/\alpha}$, $\alpha > 1$, $K > 0$, we have

$$\begin{aligned}\eta_{ki} &= \langle |G_{ji} - G_{ki}||G_{ki}\rangle \\ &= \int |x - G_{ki}| p(x)\,\mathrm{d}x \\ &= G_{ki} + \frac{2K}{\alpha-1} G_{ki}^{-(\alpha-1)} - \frac{\alpha}{\alpha-1} K^{\frac{1}{\alpha}}.\end{aligned}$$

Thus η_{ki} belongs to the attraction domains of the type II distribution in extreme value theory. Therefore

$$\max_i \eta_{ki} \sim (KN)^{1/\alpha}.$$

Hence the system is stable if

$$N < (\delta(-\langle G_{12}\rangle))^{\alpha} \cdot C$$

where C is a positive constant (see the following section).

The physical meaning of the three types of behaviour is very clear. For the global-stable case, since the variation between interactions $\frac{G_{ij}}{N}$ is small (G_{ij} bounded), we could expect that the synchronization state could be stable, independent of the system size N. For the exponential-stable and power-stable cases, the synchronization state is stable when the system is finite. Essentially, type I distribution indicates that the random interaction is exponentially distributed. The variance of interactions between units is greater than the bounded interaction case, but it is smaller than the power-stable case which corresponds to a long-tail distribution and introduces a greater variation. As a consequence, different stable conditions depending on the system size are required.

4. Examples

We now illustrate the general approach by applying the above results to one example where analytical results are possible. We consider the coupled differential equation systems with $\mathbf{H}(\mathbf{x}) = \mathbf{x}$ [21]. It is easy to see that $D\mathbf{H}$ is a $M \times M$ identity matrix. The Lyapunov exponents for equation (4) are easily calculated since the identity matrix commutes with $\mathbf{J}(\mathbf{x})$. Denoting them by $\mu_1(\lambda), \mu_2(\lambda), \ldots, \mu_M(\lambda)$, we have

$$\mu_i(\lambda) = h_i + \frac{1}{N}\mathrm{Re}(\lambda) \qquad i = 1, 2, \ldots, M. \tag{16}$$

For stability, we require the transverse Lyapunov exponents ($\lambda \neq 0$) to be negative. This is equivalent to the statement that the maximum Lyapunov exponent is less than zero:

$$\mu_{\max}(\lambda) = h_{\max} + \frac{1}{N}\mathrm{Re}(\lambda) < 0. \tag{17}$$

In other words, the stability zone Ω is the region defined by $\mathrm{Re}(\lambda) < -Nh_{\max}$. The distance function from the centre of each Gershgörin disc to the stability boundary is given by $\delta(G_{ii} - G_{ki}) = -Nh_{\max} - (G_{ii} - G_{ki})$ $(i = 1, \ldots, N, i \neq k)$. Thus the kth set of stability conditions is

$$(G_{ii} - G_{ki}) \leqslant -Nh_{\max} \tag{18}$$

$$\sum_{j=1, j\neq i}^{N} |G_{ji} - G_{ki}| \leqslant -Nh_{\max} - (G_{ii} - G_{ki}) \qquad i = 1, 2, \ldots, N \qquad i \neq k. \tag{19}$$

It is obvious that the second inequality implies the first one. So the stability condition for the synchronized state (whether an equilibrium, periodic or chaotic state) is given by

$$\sum_{j=1, j\neq i}^{N} |G_{ji} - G_{ki}| + (G_{ii} - G_{ki}) \leqslant -Nh_{\max} \qquad i = 1, 2, \ldots, N \qquad i \neq k. \tag{20}$$

When the interaction is i.i.d., by equations (20), (12) and (15), we see that

$$\min_k \max_i \langle |G_{ji} - G_{ki}||G_{ki}\rangle \leqslant -h_{\max} + (\langle G_{12}\rangle)$$

is a sufficient condition to ensure the stability of the synchronization state.

When the coupling is symmetric, i.e. $G_{ij} = G_{ji}$, Rangarajan and Ding [38, 8], based on the use of Hermitian and positive semidefinite matrices, derived a very simple stability constraint

$$G_{ij} \geqslant h_{\max} \qquad \forall\, i, j. \tag{21}$$

We assume that $N(N-1)/2$ random variables $G_{ij}, i = 1, 2, N-1, j > i$ are i.i.d. Equation (21) is reduced to

$$\min_{i<j} G_{ij} \geqslant h_{\max}$$

or equivalently

$$\max_{i<j}(-G_{ij}) \leqslant -h_{\max}. \tag{22}$$

Hence we have three different types of behaviour, as described before.

(i) *Global-stable*. For example, $-G_{ij}, i < j$ are uniformly distributed in $[a, -h_{\max}]$, where $a < -h_{\max}$.

(ii) *Exponential-stable*. For example, $-G_{ij}, i < j$ are normally distributed with mean φ and variance σ^2. We know that $\xi_{ij} = \frac{-G_{ij}-\varphi}{\sigma}$ is a standard normally random variable and by [28]

$$\langle \max_{i<j} \xi_{ij}\rangle \approx b_N + \left[\int x \exp(-\exp(-x)) \exp(-x)\,\mathrm{d}x\right] \Big/ a_N$$

where

$$\begin{cases} a_N = \sqrt{\log[N(N-1)/2]} \\ b_N = a_N - 1/2\{\log\log[N(N-1)/2] + \log 4\pi\}/a_N. \end{cases} \tag{23}$$

Thus

$$\begin{aligned}\langle \max\nolimits_{i<j}(-G_{ij})\rangle &= \sigma \max\nolimits_{i<j}(\xi_{ij}) + \varphi \\ &\approx \sigma b_N + \sigma \left[\int x \exp(-\exp(-x)) \exp(-x)\,\mathrm{d}x\right] \Big/ a_N + \varphi. \end{aligned} \tag{24}$$

It is readily seen that $b_N \sim a_N$ when N is large. Hence, by equations (23) and (24), (22) is

$$N < N_E = \frac{1 + \sqrt{1 + 8 \exp\left(\frac{-h_{\max}-\varphi}{\sigma}\right)^2}}{2}. \tag{25}$$

Now we are in the position to compare our results with results in the literature [9, 32]. In [32], May considered a linear dynamics with random interactions, i.e. $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$, $\mathbf{B} = h_{\max}\mathbf{I} + [\mathbf{A} + \Phi]/N$, where Φ is a constant matrix such that each element is equal to φ (mean), $\mathbf{I}$ the unit matrix and $\mathbf{A} = (a_{ij})$ a random symmetry matrix such that for any $i < j$, a_{ij} are normally distributed with mean zero and variance σ^2, and a_{ii} are normally distributed with mean zero and variance $2\sigma^2$. By the Weyl theorem and semicircle law [9, 5] we have

$$\begin{aligned}\lambda_{\max}(B) &\leqslant \lambda_{\max}(\mathbf{A}/N + h_{\max}\mathbf{I}) + \lambda_{\max}(\Phi)/N \\ &= \lambda_{\max}(\mathbf{A}/N + h_{\max}\mathbf{I}) + \varphi \\ &\leqslant \lambda_{\max}(\mathbf{A})/N + \varphi + h_{\max} \\ &\leqslant 2\sigma/\sqrt{N} + \varphi + h_{\max}.\end{aligned}$$

It is easily seen that if

$$N > N_{\mathrm{May}} = \left[\frac{2\sigma}{-\varphi - h_{\max}}\right]^2 \tag{26}$$

together with $-\varphi - h_{\max} > 0$, the dynamics is stable.

In figure 1 (left) we plot the stability regions obtained from the N_{May} and N_E with $\varphi = \sigma = 1$. We want to emphasize here that the two results are not completely comparable. First, the dynamical systems we consider from the beginning imply the synchronization of the uncoupled dynamics. Before coupling, there is a dynamics which could be a limit cycle, a fixed point or a chaotic attractor. We ask which kind of coupling could make the original uncoupled dynamics synchronized. In May's example, the stable regime could correspond to attractors introduced by the couplings. Secondly, in May's dynamics, only the stability of fixed points is considered. We are interested not only in fixed points, but also limit cycles or chaotic attractors. Thirdly, as a consequence of the considerations above, the requirement on the coupling matrix is different: it is required that $\sum G_{ij} = 0$ in our setup.

(iii) *Power-stable*. From the extreme value theory we conclude that

$$\langle \max\{-G_{ij}\}\rangle/\sqrt{N(N-1)/2} \approx \int \alpha x^{-\alpha} \exp(-x^{-\alpha})\,\mathrm{d}x.$$

Hence when

$$N < N_P = \frac{1 + \sqrt{4h_{\max}^2\left(\int \alpha x^{-\alpha} \exp(-x^{-\alpha})\,\mathrm{d}x\right)^{-2}}}{2} \tag{27}$$

and $h_{\max} < 0$ the system is stable. In figure 1 (right) we plot the stable regions with $\alpha = 3$ for N_P and $\varphi = \sigma = 1$ for N_E.

In figure 1 (bottom panel) we plot the stable regions (below lines) with $h_{\max} > 0$, i.e. the original map is chaotic. As one could expect, the region decreases when the interaction mean ϕ increases (bottom left). It also decreases when the variance increases.

5. Discussions

In summary, we have set up a general formalism to study the stability of synchronized states in stochastic coupled ordinary differential equations. We have also considered the often used coupling function $\mathbf{H}(\mathbf{x}) = \mathbf{x}$ for stochastic coupled ODEs and given analytical results in these cases. Three different types of behaviour are found for stochastic coupled dynamical systems.

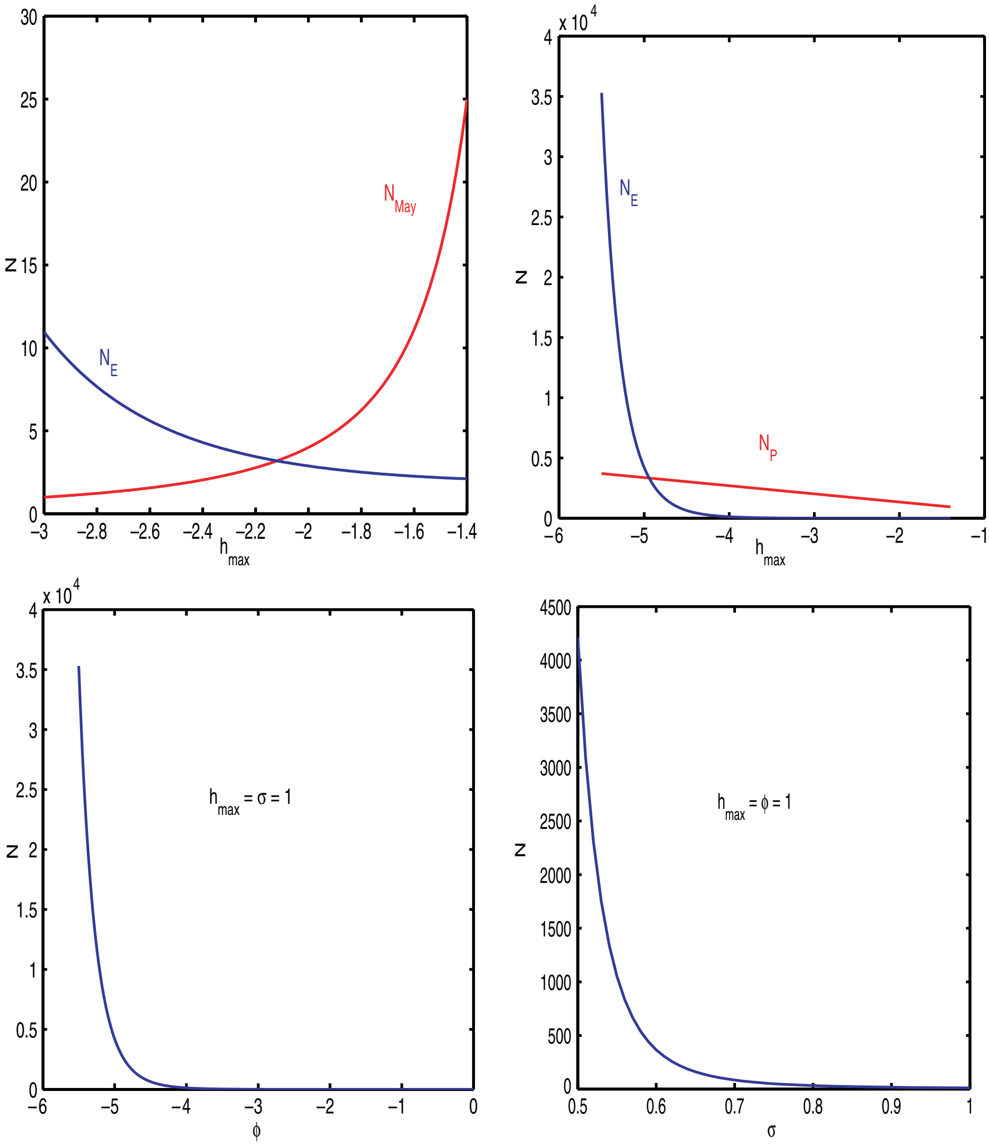

Figure 1. Top left, a comparison between the stability of N_{May} and N_E. The region above N_{May} is the stability region for N_{May} and the region below N_E is the stability region for N_E. Top right, a comparison between N_E and N_P. The regions below the curves are the stable regions. Bottom left, N_E versus ϕ with $h_{\max} = \sigma = 1$. Bottom right, N_E versus σ with $h_{\max} = \phi = 1$.

The generalization of our results to some interesting cases is almost straightforward. For example, let us consider the network of small-world connections. From equation (15), under the assumptions of finite connections of i, we can relax the max in equation (15). Hence equation (15) is reduced to finding the minimum of a sequence of random variables. We will report results on this direction in our further publications. Applications of our results to Kuramoto dynamics would be another interesting topic.

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